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HOMOTOPY GROUPS AND COHOMOTOPY GROUPS ON C_{η_n}

Howon Choi

Abstract. A map $\eta_n : S^{n+1} \to S^n$ is the generator of $\pi_{n+1}(S^n)$ and there exists a mapping cone $C_{\eta_n} = S^n \cup_{\eta_n} e^{n+2}$. The focus of this study is to clearly identify the generators of the homotopy groups and cohomotopy groups of C_{η_n} for $n \geq 3$.

1. Introduction

For any CW-complexes X and Y, the set $[X, Y]$ consists of all homotopy classes of maps from X to Y. Let S^k be the k-sphere. Then, $[S^k, Y]$ is the k-th homotopy group of Y, simply denoted by $\pi_k(Y)$ and $[X, S^k]$ is the k-th cohomotopy group(or set) of X, simply denoted by $\pi^k(X)$. For any $f : X \to Y$, there exists a mapping cone $C_f = Y \cup_f CX$. Many researchers have conducted studies related to this; [1], [3], [5], [9], [11].

In this paper, we determine the homotopy groups and cohomotopy groups of C_{η_n} where $\eta_n : S^{n+1} \to S^n$ for $n \geq 3$. In [11], $\pi_{n+2}(C_{\eta_n})$ and $\pi^n(C_{\eta_n})$ were calculated but no results for other dimensions have been provided. In previous studies, the computations were primarily conducted using algebraic tools, making it difficult to identify the generators. We clearly distinguish the generators by maps, providing a more explicit understanding of the structure. The following results have been proved.

By Theorem 1, we have:

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$$\pi_k(C_{\eta_n}) \cong \begin{cases} 0, & \text{if } k \le n-1, \\ \mathbb{Z}\ \{i_*(\iota_n)\}, & \text{if } k = n, \\ 0, & \text{if } k = n+1, \\ \mathbb{Z}\ \{\alpha_n\}, & \text{if } k = n+2. \end{cases}$$

where $\pi_*(\alpha_n) = 2\iota_{n+1}$ and $n \ge 3$.

For $n \ge 5$, by Theorem 2, we find that $\pi_{n+3}(C_{\eta_n}) \cong \mathbb{Z}_4\{i_*(\nu_n)\} \oplus \mathbb{Z}_3\{i_*(\xi)\}$.

According to Theorem 3, we have:

$$\pi^k(C_{\eta_n}) \cong \begin{cases} 0, & \text{if } k \ge n+3, \\ \mathbb{Z}\ \{\pi^*(\iota_n)\}, & \text{if } k = n+2, \\ 0, & \text{if } k = n+1, \\ \mathbb{Z}\ \{\beta_n\}, & \text{if } k = n. \end{cases}$$

where $i^*(\beta_n) = 2\iota_{n+1}$ and $n \ge 3$.

For $n \ge 6$, by Theorem 4, we have $\pi^{n-1}(C_{\eta_n}) \cong \mathbb{Z}_4\{\pi^*(\nu_{n-1})\} \oplus \mathbb{Z}_3\{\pi^*(\xi_{n-1})\}$.

Throughout this note, all (topological) spaces are have homotopy type based CW-complexes, and all maps and homotopies preserve the base point. For given spaces X and Y, we denote by $[X, Y]$ the set of (based) homotopy classes of maps of X to Y, and by the same letter f a map $f : X \to Y$ and its homotopy class $f \in [X, Y]$. Also, we denote usually by

$$f_* : [Z, X] \to [Z, Y],\ g^* : [Y, Z] \to [X, Z]$$

for any Z. If a group G is generated by a set $\{a_1, a_2, ..., a_n\}$, then we denoted the group by $G\{a_1, a_2, ..., a_n\}$.

2. Preliminaries

Let X be a space. Then we denote by SX the suspension of X and by S^nX the iterated suspension defined by $S^nX = S(S^{n-1}X)$. For a map $f : A \to B$, there is a mapping cone $C_f = B \cup_f CA$ of f. Then we have the mapping cone sequence:

$$A \xrightarrow{f} B \xrightarrow{i} C_f \xrightarrow{\pi} SA \xrightarrow{Sf} SB \xrightarrow{Si} SC_f \xrightarrow{s\pi} S^2A \longrightarrow \cdots .$$

By [8], the following sequence is exact for any space X:

$$\cdots \longrightarrow [SB, X] \xrightarrow{Sf^*} [SA, X] \xrightarrow{\pi^*} [C_f, X] \xrightarrow{i^*} [B, X] \xrightarrow{f^*} [A, X].$$

By [4], if A is an m-connected and B is an n-connected, then we have the following exact sequence for any Y with dimension at most $m+n$:

$$[Y, A] \xrightarrow{f_*} [Y, B] \xrightarrow{i_*} [Y, C_f] \xrightarrow{\pi_*} [Y, SA] \xrightarrow{Sf_*} [Y, SB] \longrightarrow \cdots .$$

By [12], the generators of some homotopy groups of spheres can be summarized as follows:

	$i<0$	$i=0$	$i=1$	$i=2$
$[S^{n+i}, S^n]$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
Generator		ι_n	η_n	η_n^2

for $n \geq 3$. And $\pi_{n+3}(S^n) \cong \mathbb{Z}_8\{\nu_n\} \oplus \mathbb{Z}_3\{\xi_n\}$ for $n \geq 5$.

3. The homotopy groups on C_{η_n} for $n \geq 3$.

Let S^k be the k-dimensional sphere. There exists a mapping cone sequences for $\eta_n : S^{n+1} \to S^n$:

$$S^{n+1} \xrightarrow{\eta_n} S^n \xrightarrow{i} C_{\eta_n} \xrightarrow{\pi} S^{n+2} \xrightarrow{\eta_{n+1}} S^{n+1} \xrightarrow{Si} \cdots$$

where $C_{\eta_n} = S^n \cup_{\eta_n} e^{n+2}$ and $n \geq 3$. In this section, we find generators of homotopy groups on C_{η_n}.

By [4], we obtain the long exact sequence:

$$[S^k, S^{n+1}] \xrightarrow{\eta_{n*}} [S^k, S^n] \xrightarrow{i_*} [S^k, C_{\eta_n}] \xrightarrow{\pi_*} [S^k, S^{n+2}] \xrightarrow{\eta_{n+1*}} [S^k, S^{n+1}] \longrightarrow \cdots$$

for $k \leq 2n-1$.

Case 1. $k \leq n-1$.
We have

$$0 \xrightarrow{i_*} [S^k, C_{\eta_n}] \xrightarrow{\pi_*} 0.$$

Hence $\pi_k(C_{\eta_n}) \cong 0$ for $k \leq n-1$.

Case 2. $k = n$.
We have

$$0 \xrightarrow{\eta_{n*}} \mathbb{Z}\{\iota_n\} \xrightarrow{i_*} [S^n, C_{\eta_n}] \xrightarrow{\pi_*} 0.$$

Hence $\pi_n(C_{\eta_n}) \cong \mathbb{Z}\{i_*(\iota_n)\}$.

Case 3. $k = n+1$.
We have

$$\mathbb{Z}\{\iota_{n+1}\} \xrightarrow{\eta_{n*}} \mathbb{Z}_2\{\eta_n\} \xrightarrow{i_*} [S^{n+1}, C_{\eta_n}] \xrightarrow{\pi_*} 0.$$

Since $\eta_{n*}(\iota_{n+1}) = \eta_n \circ \iota_{n+1} = \eta_n$ and so the i_* is the trivial homomorphism,

$$0 \xrightarrow{i_*} [S^{n+1}, C_{\eta_n}] \xrightarrow{\pi_*} 0.$$

Hence $\pi_{n+1}(C_{\eta_n}) \cong 0$.

Case 4. $k = n+2$.
We have

$$\mathbb{Z}_2\{\eta_{n+1}\} \xrightarrow{\eta_{n*}} \mathbb{Z}_2\{\eta_n^2\} \xrightarrow{i_*} [S^{n+2}, C_{\eta_n}] \xrightarrow{\pi_*} \mathbb{Z}\{\iota_{n+2}\} \xrightarrow{\eta_{n+1*}} \mathbb{Z}_2\{\eta_{n+1}\} \longrightarrow \cdots.$$

Since $\eta_{n*}(\eta_{n+1}) = \eta_n \circ \eta_{n+1} = \eta_n^2$, the i_* is the trivial homomorphism. Since $\eta_{n+1*}(\iota_{n+2}) = \eta_{n+1} \circ \iota_{n+2} = \eta_{n+1}$, the π_* is an injective. It implies that

$$0 \xrightarrow{i_*} [S^{n+2}, C_{\eta_n}] \xrightarrow{\pi_*} \mathbb{Z}\{\iota_{n+2}\} \xrightarrow{\eta_{n+1*}} \mathbb{Z}_2\{\eta_{n+1}\} \longrightarrow 0.$$

Hence $\pi_{n+2}(C_{\eta_n}) \cong \mathbb{Z}\{\alpha_n\}$ such that $\pi_*(\alpha_n) = 2\iota_{n+1}$.

We can prove the following theorem according to cases 1, 2, 3, and 4.

THEOREM 1.

$$\pi_k(C_{\eta_n}) \cong \begin{cases} 0, & \text{if } k \leq n-1, \\ \mathbb{Z}\ \{i_*(\iota_n)\}, & \text{if } k = n, \\ 0, & \text{if } k = n+1, \\ \mathbb{Z}\ \{\alpha_n\}, & \text{if } k = n+2. \end{cases}$$

where $\pi_*(\alpha_n) = 2\iota_{n+1}$ *and* $n \geq 3$.

THEOREM 2. *For* $n \geq 5$,

$$\pi_{n+3}(C_{\eta_n}) \cong \mathbb{Z}_4\{i_*(\nu_n)\} \oplus \mathbb{Z}_3\{i_*(\xi)\}$$

Proof. By [4], we obtain the long exact sequence:

$$\mathbb{Z}_2\{\eta_{n+1}^2\} \xrightarrow{\eta_{n*}} \mathbb{Z}_8\{\nu_n\} \oplus \mathbb{Z}_3\{\xi_n\} \xrightarrow{i_*} [S^{n+3}, C_{\eta_n}] \xrightarrow{\pi_*} \mathbb{Z}_2\{\eta_{n+2}\} \xrightarrow{\eta_{n+1*}} \mathbb{Z}_2\{\eta_{n+1}^2\} \longrightarrow \cdots$$

Since $\eta_{n*}(\eta_{n+1}^2) = \eta_n \circ \eta_{n+1}^2 = \eta_n^3$, $\eta_n^3 = 4\nu_n$ by [12]. Since $\eta_{n+1*}(\eta_{n+2}) = \eta_{n+1} \circ \eta_{n+2} = \eta_{n+1}^2$, the π_* is the trivial homomorphism. By the exactness, we have

$$0 \xrightarrow{\eta_{n*}} \mathbb{Z}_4 \oplus \mathbb{Z}_3 \xrightarrow{i_*} [S^{n+3}, C_{\eta_n}] \xrightarrow{\pi_*} 0$$

Hence $\pi_{n+3}(C_{\eta_n}) \cong \mathbb{Z}_4\{i_*(\nu_n)\} \oplus \mathbb{Z}_3\{i_*(\xi_n)\}$. □

4. The co-homotopy groups on C_{η_n} for $n \geq 3$.

By [8], we obtain the long exact sequence:

$$\cdots \longrightarrow [S^{n+1}, S^k] \xrightarrow{\eta_{n+1}^*} [S^{n+2}, S^k] \xrightarrow{\pi^*} [C_{\eta_n}, S^k] \xrightarrow{i^*} [S^n, S^k] \xrightarrow{\eta_n^*} [S^{n+1}, S^k].$$

Case 1. $k \geq n+3$.
We have

$$\cdots \longrightarrow 0 \xrightarrow{\eta_{n+1}^*} 0 \xrightarrow{\pi^*} [C_{\eta_n}, S^k] \xrightarrow{i^*} 0 \xrightarrow{\eta_n^*} 0.$$

Hence $\pi^k(C_{\eta_n}) \cong 0$ for $k \geq n+3$.

Case 2. $k = n+2$.
We have

$$\cdots \longrightarrow 0 \xrightarrow{\eta_{n+1}^*} \mathbb{Z}\{\iota_{n+2}\} \xrightarrow{\pi^*} [C_{\eta_n}, S^{n+2}] \xrightarrow{i^*} 0 \xrightarrow{\eta_n^*} 0.$$

Hence $\pi^{n+2}(C_{\eta_n}) \cong \mathbb{Z}\{\pi^*(\iota_{n+2})\}$.

Case 3. $k = n+1$.
We have

$$\cdots \longrightarrow \mathbb{Z}\{\iota_{n+1}\} \xrightarrow{\eta_{n+1}^*} \mathbb{Z}_2\{\eta_{n+1}\} \xrightarrow{\pi^*} [C_{\eta_n}, S^{n+1}] \xrightarrow{i^*} 0.$$

Since $\eta_{n+1}^*(\iota_{n+1}) = \eta_{n+1}$, the π^* is the trivial homomorphism. It implies that

$$0 \xrightarrow{\pi^*} [C_{\eta_n}, S^{n+1}] \xrightarrow{i^*} 0.$$

Hence $\pi^{n+1}(C_{\eta_n}) \cong 0$.

Case 4. $k = n$.
We have

$$\cdots \longrightarrow \mathbb{Z}_2\{\eta_n\} \xrightarrow{\eta^*_{n+1}} \mathbb{Z}_2\{\eta^2_n\} \xrightarrow{\pi^*} [C_{\eta_n}, S^n] \xrightarrow{i^*} \mathbb{Z}\{\iota_n\} \xrightarrow{\eta^*_n} \mathbb{Z}_2\{\eta_n\}.$$

Since $\eta^*_{n+1}(\eta_n) = \eta^2_n$, the π^* is the trivial homomorphism. It implies that

$$0 \xrightarrow{\pi^*} [C_{\eta_n}, S^n] \xrightarrow{i^*} \mathbb{Z}\{\iota_n\} \xrightarrow{\eta^*_n} \mathbb{Z}_2\{\eta_n\}.$$

Since $\eta^*_n(\iota_n) = \eta_n$, the η^*_n is the surjective homomorphism.
Hence $\pi^{n+1}(C_{\eta_n}) \cong \mathbb{Z}\{\beta_n\}$ such that $i^*(\beta_n) = 2\iota_n$.

We can prove the following theorem according to cases 1, 2, 3, and 4.

THEOREM 3.

$$\pi^k(C_{\eta_n}) \cong \begin{cases} 0, & \text{if } k \geq n+3, \\ \mathbb{Z}\ \{\pi^*(\iota_n)\}, & \text{if } k = n+2, \\ 0, & \text{if } k = n+1, \\ \mathbb{Z}\ \{\beta_n\}, & \text{if } k = n. \end{cases}$$

where $i^*(\beta_n) = 2\iota_{n+1}$ *and* $n \geq 3$.

THEOREM 4. *For* $n \geq 6$,

$$\pi^{n-1}(C_{\eta_n}) \cong \mathbb{Z}_4\{\pi^*(\nu_{n-1})\} \oplus \mathbb{Z}_3\{\pi^*(\xi_{n-1})\}$$

Proof. By [8], we obtain the long exact sequence:

$$\mathbb{Z}_2\{\eta^2_{n-1}\} \xrightarrow{\eta^*_{n+1}} \mathbb{Z}_8\{\nu_{n-1}\} \oplus \mathbb{Z}_3\{\xi_{n-1}\} \xrightarrow{\pi^*} [C_{\eta_n}, S^{n-1}] \xrightarrow{i^*} \mathbb{Z}_2\{\eta_{n-1}\} \xrightarrow{\eta^*_n} \mathbb{Z}_2\{\eta^2_{n-1}\}.$$

By the exactness, we have

$$0 \xrightarrow{\eta^*_{n+1}} \mathbb{Z}_4 \oplus \mathbb{Z}_3 \xrightarrow{\pi^*} [C_{\eta_n}, S^{n-1}] \xrightarrow{i^*} 0.$$

Hence $\pi^{n-1}(C_{\eta_n}) \cong \mathbb{Z}_4\{\pi^*(\nu_{n-1})\} \oplus \mathbb{Z}_3\{\pi^*(\xi_{n-1})\}$. □

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Howon Choi
Faculty of Liberal Arts and Teaching
Kangnam University
40 Gangnam-ro, Yongin-si, 16979, Korea
E-mail: howon@kangnam.ac.kr